Mordell-Weil ranks of Jacobians of isotrivial families of plane curves

Remke Kloosterman

Humboldt-Universität zu Berlin

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Theorem (Hulek–K., 2008)

Let $n$ be a positive integer. Consider
\[ U = \{(A, B) \in \overline{\mathbb{Q}}[s, t]_{\leq 4n} \oplus \overline{\mathbb{Q}}[s, t]_{\leq 6n} \mid 4A^3 + 27B^2 \neq 0\}. \]
Let $C \subset U$ be the subset such that the Mordell-Weil rank of
\[ E_{A,B} : y^2 = x^3 + A(s, t)x + B(s, t) \]
over $\overline{\mathbb{Q}}(s, t)$ is effectively computable. Then $C$ is dense in $U$. 
Proof: Take $V \subset U$ to be the subset of pairs $(A, B)$ such that the corresponding curves in $\mathbb{P}^2$ intersect in precisely $24n^2$ distinct points.

Then for every $(A, B) \in V$ we have that $E_{A, B}(C(s, t)) = \{O\}$.

Hence $V \subset C$.

However, the codimension of $U \setminus V$ in $U$ is one, and the codimension of $U \setminus C$ in $U$ tends to infinity for $n \to \infty$.

The fun is happening in $C \setminus V$. 
Introduction

Theorem (Cogolludo-Agustin–Libgober, 2010)

- Let $n$ be a positive integer. Let $g \in \mathbb{C}[y_0, y_1, y_2]_{6n}$ be a squarefree homogeneous polynomial.
- Suppose that $\Delta := V(g) \subset \mathbb{P}^2$ is a curve with only nodes and ordinary cusps as singularities. Let $\Sigma \subset \mathbb{P}^2$ be the set of cusp of $\Delta$.
- Let $I$ be the ideal of $\Sigma$, and set

$$\delta := \#\Sigma - \dim(\mathbb{C}[y_0, y_1, y_2]/I)_{5n-3}.$$

(The defect of the linear system of degree $5n - 3$ polynomials through $\Sigma$.)

Then the Mordell-Weil rank of $y^2 = x^3 + g(s, t, 1)$ over $\mathbb{C}(s, t)$ equals $2\delta$. 
Introduction

There are two proofs for this result.
One uses Albanese varieties. (Cogolludo-Agustin–Libgober, Libgober)
One uses a generalized Thom-Sebastiani result. (K.)
Both approaches generalize completely differently.
Both approaches have some very nice applications.
Let $\mathcal{A} \to \mathbb{P}^2$ be an isotrivial family of abelian varieties, with discriminant $\Delta$. Let $A/C(s, t)$ be the generic fiber.

Then there exist

- a (singular) projective surface $S$, admitting a finite Galois cover $\pi : S \to \mathbb{P}^2$ with group $G$, ramified only above $\Delta$,
- an abelian variety $A_0/C$ admitting a $G$-action and
- a $G$-equivariant resolution of singularities $\tilde{S}$ of $S$

such that the fibration $(A_0 \times \tilde{S})/G \to \tilde{S}/G$ is birational to $\mathcal{A} \to \mathbb{P}^2$.

The map $\tilde{S} \to \mathbb{P}^2$ is called the *trivializing base change*. 
Diagram

\[
\begin{array}{c}
\tilde{S} \times A \\
\downarrow \\
\tilde{S} \\
\end{array} \quad \begin{array}{c}
\rightarrow A \\
\downarrow \\
S \\
\rightarrow \mathbb{P}^2 \\
\end{array}
\]
The points in $A(C(s, t))$ correspond with *rational sections* $P^2 \rightarrow A$.

The rational sections $P^2 \rightarrow A$ corresponds with $G$-equivariant rational sections $\tilde{S} \rightarrow \tilde{S} \times A_0$.

The latter correspond with graphs of $G$-equivariant rational maps $\tilde{S} \rightarrow A_0$.

Since the target is an abelian variety, we can extend each such a rational map to a morphism $\tilde{S} \rightarrow A_0$.

Hence it sufficies to study the possible morphisms $\text{Alb}(\tilde{S}) \rightarrow A_0$.

**Difference with one-dimensional base variety:** $\text{Alb}(\tilde{S})$ is controlable!
Diagram

\[
\begin{array}{ccc}
\tilde{S} \times A & \rightarrow & A \\
\downarrow & & \downarrow \\
\tilde{S} & \rightarrow & S & \rightarrow & \mathbb{P}^2
\end{array}
\]
**Albanese approach**

- **Difference with one-dimensional base variety:** \(\text{Alb}(\tilde{S})\) is controllable!

- Assume (for simplicity) that \(\Delta\) is reduced.

- For each \(p \in \Delta_{\text{sing}}\) there is a so-called local Albanese variety \(\text{Alb}_p\). (E.g., if \(p\) is a cusp then \(\text{Alb}_p\) is the \(j = 0\) elliptic curve, if \(p\) is a node then \(\text{Alb}_p = 0\).)

- **Theorem (Libgober’s local divisibility):** \(\text{Alb}(\tilde{S})\) is an isogeny factor of \(\prod_{p \in \Delta_{\text{sing}}} \text{Alb}_p\).
Theorem (Libgober’s local divisibility): \( \text{Alb}(\tilde{S}) \) is an isogeny factor of \( \prod_{p \in \Delta_{\text{sing}}} \text{Alb}_p \).

In the case of \( y^2 = x^3 + g(s, t, 1) \), \( \Delta = V(g) \) is a cuspidal plane curve we have that \( \text{Alb}_p = 0 \) or \( \text{Alb}_p \cong E_0 \).

Hence \( \text{Alb}(\tilde{S}) \) is isogeneous to a power of \( E_0 \).

We obtain that the Mordell-Weil rank equals \( 2 \dim \text{Alb}(\tilde{S}) \).
Albanese approach (proof of C–L)

- **Theorem (Zariski–Libgober):** If $G$ is cyclic then $\dim \text{Alb}(\tilde{S})$ is effectively computable.
- Actually you find a formula in terms of defects of several very explicit linear systems.
- In the C–L case you find that $\dim \text{Alb}(\tilde{S})$ equals the defect of the linear system of degree $5/6 \deg(\Delta) - 3$ polynomials through $\Sigma$. 
Theorem (Zariski)

Let $f \in \mathbb{C}[s, t]$ be an irreducible polynomial, $m = p^n$ be a prime power. Then the Albanese variety of the desingularization of the projective closure of

$$z^m = f(s, t)$$

is trivial.
Corollary

Let \( f \in \mathbb{C}[s, t] \) be an irreducible polynomial, let \( A, B \in \mathbb{C} \) be such that \( 4A^3 + 27B^2 \neq 0 \). Let

- \( E_1 : y^2 = x^3 + f^2 \)
- \( E_2 : y^2 = x^3 + f^4 \)
- \( E_3 : y^2 = x^3 + fx \)
- \( E_4 : y^2 = x^3 + Af^2 x + Bf^3 \)

Then the rank of \( E_i(\mathbb{C}(s, t)) \) is zero.

Remark: The corollary holds also true if we replace \((\mathbb{C}[s, t], \mathbb{C}, \mathbb{C}(s, t))\) with \((K[t], K, K(t))\) and \( K \) is an algebraically closed field, but is false if \( K \) does not contain a third root of unity or a fourth root of unity.
Albanese approach

- Exploits that we can control both the dimension and the isogeny factors of the Albanese variety of the trivializing base change.
- Is very good to bound the Mordell-Weil rank.
- If you want to determine the ranks then you have to put few constraints on the general fiber $A_0$, but strong constraints on the singularities of the discriminant.
Cogolludo-Augustin–Libgober gave a formula for the Mordell-Weil rank of $y^2 = x^3 + g(s, t, 1)$, with $g$ a homogeneous polynomial such that the curve $\Delta := V(g)$ has only nodes and cusps.

We are going to generalize this as follows. We take

- a weighted homogeneous polynomial $f(x_1, x_2)$, with weights $w_1, w_2 \in \mathbb{Q}$ such that the weighted degree of $f$ equals one and such that the curve $f = 0$ in $\mathbb{C}^2$ has an isolated singularity at the origin. (E.g., $f = x_1^3 - x_2^2$, $w_1 = 1/3, w_2 = 1/2$.)
- a homogeneous polynomial $g(y_0, y_1, y_2) = 0$ such that the projective curve $V(g) \subset \mathbb{P}^2$ has isolated singularities.

Let $H/\mathbb{C}(s, t)$ be the desingularization of the projective closure of the affine curve $f(x_1, x_2) + g(s, t, 1) = 0$.

We want to determine sufficient conditions on $f, g$ such that we can compute the rank of $\text{Jac}(H)(\mathbb{C}(s, t))$. 

Thom-Sebastiani in C-L case
Affine Milnor fiber

- For the moment let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be an arbitrary weighted homogeneous polynomial.
- Choose the weights $w_1, \ldots, w_n$ such that the degree of $f$ equals 1.
- Denote with $F = \{f = 1\} \subset \mathbb{C}^n$ (smooth affine hypersurface). The affine Milnor fiber of $f$.
- For $\alpha \in \mathbb{C}$ let $\zeta(\alpha) := \exp(2\pi i \alpha)$.
- Let $T_f : \mathbb{C}^n \to \mathbb{C}^n$ be the automorphism mapping $x_i \to \zeta(w_i)x_i$.
- Then $T_f(F) = F$. Hence $T_f$ induces a operator on $H^\bullet(F)$. 
Let $s$ be the dimension of the singular locus of $f = 0$ in $\mathbb{C}^n$.

Then $\tilde{H}^i(F) = 0$ for $i \geq n$ and $i \leq n - 2 - s$.

So in our setup ($f(x_1, x_2) = 0$ has an isolated singularities and $g(y_0, y_1, y_2) = 0$ has one-dimensional singular locus) we have three intersecting groups $H^1(F), H^1(G), H^2(G)$. 
We required that \( f \) has an isolated singularity. For a general weighted homogeneous isolated singularity in \( n \) variables there is a formula for the dimension of the \( \zeta(\alpha) \) eigenspace of \( T_f \) acting on \( H^{n-1}(F) \) in terms of the Jacobian ring of \( f \).

In our case: For \( \alpha \in \mathbb{Q} \) set

\[
\nu(\alpha) := \dim(\mathbb{C}[x_1, x_2]/(f_{x_1}, f_{x_2}))_{\alpha+1-w_1-w_2}
\]

Multiplicity of \( \alpha \) in the Steenbrink spectrum of \( f \).

\( \nu(\alpha) \) only depends on \( w_1, w_2 \) and \( \alpha \).

Symmetry: \( \nu(-\alpha) = \nu(\alpha) \).

For \( 0 \leq \alpha < 1 \) we have that \( \dim H^{n-1}(F)_{\zeta(\alpha)} \) equals \( \nu(\alpha) + \nu(\alpha - 1) \).
\( H^1(G) \)

- \( H^1(G) \) depends on the singularities of \( C \) and their position.
- Libgober studied ideals of quasi-adjunction (multiplier ideals). He gave for any \( 0 < \alpha < 1 \) an effective construction of schemes \( X^{(\alpha)} \subset \Delta_{\text{sing}} \) such that if

\[
\delta_\alpha := \text{length}(X^{(\alpha)}) - \dim \mathbb{C}[y_0, y_1, y_2]/(I(X^{(\alpha)}))_{\alpha d - 3}
\]

Then

\[
\dim H^1(G)_{\zeta(\alpha)} = \delta(\alpha) + \delta(1 - \alpha)
\]

- \( \dim H^1(G)_{\zeta(0)} \) equals the number of irreducible components of \( \Delta \) minus one.
Where does this come from:

- You may know Alexander polynomial of knots.
- Can also be defined for plane curves (Zariski).
- This is defined in terms of the fundamental group of $\mathbb{P}^2 \setminus \Delta$.
- Equivalent definition of the Alexander polynomial of $G$ is the characteristic polynomial of $T_g$ on $H^1(G)$.
- So the above tells you also how to compute the Alexander polynomial of a plane curve.
Main result

Theorem

Let $f \in \mathbb{C}[x_1, x_2]$ be a weighted homogeneous polynomial with rational weights $w_1, w_2$ and of weighted degree 1 and let $g \in \mathbb{C}[y_0, y_1, y_2]$ be a squarefree homogeneous polynomial of degree $d$. Assume

$$dw_1, dw_2 \in \mathbb{Z} \quad \text{and} \quad \sum_{0 \leq \alpha < 1} \nu(\alpha)\delta_\alpha = 0.$$  

Then the Mordell-Weil rank of the group of $\mathbb{C}(s, t)$-valued points of the Jacobian of the general fiber of $H : f(x, y) + g(s, t, 1)$ equals

$$\sum_{0 < \alpha < 1} (\nu(\alpha) + \nu(\alpha - 1))(\delta(\alpha) + \delta(1 - \alpha)).$$
Main result

If $f = x_1^2 + x_2^e$ and $V(g)$ is a curve with ADE singularities then

$$
\sum_{0 \leq \alpha < 1} \nu(\alpha)\delta_{\alpha} = 0.
$$

If $f = x_1^2 + x_2^3$ and $V(g)$ is a curve with nodes and ordinary cusps then you recover C–L.
Proof: reduction to a problem on affine Milnor fibers

- $W : Z(f(x_1, x_2) + g(y_0, y_1, y_2)) \subset \mathbb{P}(w_1, w_2, 1, 1, 1)$
- $H/C(s, t)$ the smooth projective curve associated with $f(x_1, x_2) + g(s, t, 1)$.
- A variant of Shioda-Tate formula yields

$$\text{rank } \text{Jac}(H)(C(s, t)) = \text{rank } \text{CH}^1(W) - \delta(0)\nu(0) - 1$$
Cycle class maps for singular varieties

- $\text{CH}^1(W) \otimes \mathbb{Q}$ can be mapped injectively into $H^4(W, \mathbb{Q})$. (Lefschetz $(1, 1)$ is unavailable, however every Weil divisor has a fundamental class inside $H_4(W, \mathbb{Z})$.)
- The image is sub-MHS of pure type $(2, 2)$.
- If $H^4(W, \mathbb{Q})$ is of pure $(2, 2)$ type then $\text{CH}^1(W) \otimes \mathbb{Q} \cong H^4(W, \mathbb{Q})$.
- Let $F \oplus G := \{f + g = 1\} \subset \mathbb{C}^5$. Then there is a natural isomorphism of MHS
  \[ H^4(W, \mathbb{Q})_{\text{prim}} \cong H^3(F \oplus G) T_{f+g} \]
- We need to find out when $H^3(F \oplus G)$ is of pure $(2, 2)$-type, and to have a way to calculate its dimension.
Thom-Sebastiani

- Advantage of working with Milnor fibers:
- Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ and $g \in \mathbb{C}[y_1, \ldots, y_m]$ weighted homogeneous.
- Let $F := \{f = 1\}$, $G := \{g = 1\}$ and $F \oplus G := \{f + g = 1\}$.
- Thom-Sebastiani:
  \[
  \tilde{H}^{\bullet+1}(F \oplus G) \cong \tilde{H}^{\bullet}(F) \otimes \tilde{H}^{\bullet}(G)
  \]
  and $T_{f+g} = T_f \otimes T_g$.
- There is no such results for $f = 0$, $g = 0$ and $f + g = 0$.
- This is not a morphism of MHS.
Thom-Sebastiani

- In our case \((f(x_1, x_2)\) has an isolated singularity, \(g(y_0, y_1, y_2)\) one-dimensional singular locus) we find

\[
H^3(F \oplus G)^{T_f+g} = \bigoplus \alpha H^1(F)_{\zeta(\alpha)} \otimes H^1(G)_{\zeta(-\alpha)}
\]

- Combining everything we get

\[
h^4(W) = 1 + \sum_{0 \leq \alpha < 1} (\nu(\alpha) + \nu(\alpha - 1))(\delta(1 - \alpha) + \delta(\alpha))
\]

- What about the MHS?
The Thom-Sebastiani isomorphism

\[ \tilde{H}^{1}(F \oplus G) \cong \tilde{H}^{1}(F) \otimes \tilde{H}^{1}(G) \]

is not an isomorphism of MHS.

If \( f = 0 \) and \( g = 0 \) are arbitrary isolated singularities then Scherk and Steenbrink related the MHS on the LHS with the one on the RHS. (Tedious formulae.)

Main point in the proof is now that we have similar formulae if \( g = 0 \) has a nonisolated singularity, but is weighted homogeneous.
**MHS**

- **Upshot:** $H^4(W)$ has a pure weight 4 MHS.
- $h^{4,0}$ and $h^{0,4}$ vanish by dimension reasons.
- $h^{3,1}$ equals
  
  $$
  \sum_{0<\alpha<1} \delta(\alpha)\nu(\alpha)
  $$

- And this vanishes by assumption. Hence
  
  $$
  h^4(W) = \text{rank} \text{Jac}(H)(C(s, t)) + \nu(0)\alpha(0) + 1
  $$

  $$
  \text{rank} \text{Jac}(H)(C(s, t)) = \sum_{0<\alpha<1} (\nu(\alpha)+\nu(\alpha-1))(\delta(1-\alpha)+\delta(\alpha))
  $$
Toric decompositions

- Let \( g \in \mathbb{C}[y_0, y_1, y_2] \) be a squarefree homogeneous polynomial.

- A \((p, q)\)-toric decomposition consists of homogeneous polynomials \( h_1, h_2 \) such that

  \[
  h_1^p + h_2^q = g
  \]

- A \((p, q)\)-quasi-toric decomposition consists of homogeneous polynomials \( h_1, h_2, h_3 \) such that

  \[
  h_1^p + h_2^q = h_3^r g
  \]

  with \( r = \text{lcm}(p, q) \).

- Equivalently, these decompositions correspond with \( \mathbb{C}[s, t] \)
  and \( \mathbb{C}(s, t) \) points on the curve

  \[
  H_{p, q} : x_1^p + x_2^q - g(s, t, 1) = 0
  \]
**Toric decompositions**

- If \((p, q) \in \{(2, 3), (3, 3), (2, 4)\}\) then \(H_{p,q} \cong \text{Jac}(H_{p,q})\).

- The quasi-toric decompositions form an abelian group and we can often calculate its rank.

- Extra structure: we can introduce a height pairing on the quasi-toric decompositions. (Shioda’s constructions of the Mordell-Weil lattice.)
Toric decompositions

- From now on \((p, q) = (2, 3), H = H_{2,3}\) and \(k = \lceil \deg(g)/6 \rceil\).
- In our case the Mordell-Weil lattice is an integral lattice.
- The nonzero vectors have length at least \(2k\). \((g\) squarefree.)
- The vectors of length \(2k\) correspond with the toric decomposition.
Irreducible sextics

- Let $g$ be an irreducible homogeneous polynomial of degree 6.
- Then $\delta(\alpha) = 0$ for $\alpha \neq 5/6$. (Zariski’s theorem excludes $1/3, 1/2, 2/3$, semicontinuity of the spectrum excludes $1/6$.)
- Degtyarev: Let $C$ be a plane sextic. Let $t$ be the number of toric decompositions of $C$. Then $(\delta(5/6), t) \in \{(0, 0), (1, 6), (2, 24), (3, 72)\}$.
- Proof uses a lot of properties of the $K3$-surface $z^2 = g(y_0, y_1, y_2)$. 
Irreducible sextics

- The toric decompositions generate a root lattice \((k = 1)\) contained in the Mordell-Weil lattice (which has rank \(2\delta(5/6)\)).
- Classification of roots lattices it follows that in each case there is only one possible lattice.
- The toric decompositions generate (depending on \(\delta(5/6)\)) the lattice 0, \(A_2, D_4, E_6\).
- In particular, \(H(\mathbb{C}(s, t))\) is generated by \(H(\mathbb{C}[s, t])\).
Zariski pairs

- Aim: find a $g \in \mathbb{C}[y_0, y_1, y_2]_{6n}$ such that the Morell-Weil group is generated (as $\mathbb{Z}[\omega]$-module) by a point of the form
  \[
  \left( \frac{h_0}{l^2}, \frac{h_1}{l^3} \right)
  \]
  with $\deg(l) = 1, \deg(h_0) = 6, \deg(h_1) = 9$.
- If $n = 1$ then the Mordell-Weil rank equals 4.
- If $n = 2$ then one can find such a $g$ with 30 cusps.
- There is also a degree 12 cuspidal curve with 30 cusps, such that the Mordell-Weil group is generated by a point of the form $(h_2, h_3)$ with $\deg(h_2) = 4, \deg(h_3) = 6$.
- They yield distinct Mordell-Weil lattices.
Can show that (for this type of curves) the Mordell-Weil lattices is a deformation invariant.

So the locus of the degree 12 curves with 30 cusps has at least 2 irreducible components.

These two curves form a so-called (Alexander-equivalent) Zariski pair.