Mordell-Weil ranks of Jacobians of isotrivial families of plane curves

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Introduction

Theorem (Hulek–K., 2008)

Let n be a positive integer. Consider
$U = \{(A, B) \in \overline{\mathbf{Q}}[s, t]_{\leq 4n} \oplus \overline{\mathbf{Q}}[s, t]_{\leq 6n} \mid 4A^3 + 27B^2 \neq 0\}$.
Let $C \subset U$ be the subset such that the Mordell-Weil rank of

$$E_{A,B} : y^2 = x^3 + A(s, t)x + B(s, t)$$

over $\overline{\mathbf{Q}}(s, t)$ is effectively computable. Then C is dense in U.

Introduction

- Proof: Take $V \subset U$ to be the subset of pairs (A, B) such that the corresponding curves in $\mathbf{P}^2$ intersect in precisely $24n^2$ distinct points.
- Then for every $(A, B) \in V$ we have that $E_{A,B}(\mathbf{C}(s, t)) = \{O\}$.
- Hence $V \subset C$.
- However, the codimension of $U \setminus V$ in U is one, and the codimension of $U \setminus C$ in U tends to infinity for $n \to \infty$.
- The fun is happening in $C \setminus V$.

Introduction

Theorem (Cogolludo-Agustin–Libgober, 2010)

- *Let n be a positive integer. Let $g \in \mathbf{C}[y_0, y_1, y_2]_{6n}$ be a squarefree homogeneous polynomial.*
- *Suppose that $\Delta := V(g) \subset \mathbf{P}^2$ is a curve with only nodes and ordinary cusps as singularities. Let $\Sigma \subset \mathbf{P}^2$ be the set of cusp of Δ.*
- *Let I be the ideal of Σ, and set*

$$\delta := \#\Sigma - \dim(\mathbf{C}[y_0, y_1, y_2]/I)_{5n-3}.$$

(The defect of the linear system of degree $5n-3$ polynomials through Σ.)

Then the Mordell-Weil rank of $y^2 = x^3 + g(s,t,1)$ over $\mathbf{C}(s,t)$ equals 2δ.

Introduction

- There are two proofs for this result.
- One uses Albanese varieties. (Cogolludo-Agustin–Libgober, Libgober)
- One uses a generalized Thom-Sebastiani result. (K.)
- Both approaches generalize completely differently.
- Both approaches have some very nice applications.

Albanese approach

- Let $\mathcal{A} \to \mathbf{P}^2$ be an isotrivial family of abelian varieties, with discriminant Δ. Let $A/\mathbf{C}(s,t)$ be the generic fiber.
- Then there exist
  - a (singular) projective surface S, admitting a finite Galois cover $\pi : S \to \mathbf{P}^2$ with group G, ramified only above Δ,
  - an abelian variety $A_0/\mathbf{C}$ admitting a G-action and
  - a G-equivariant resolution of singularities $\tilde{S}$ of S

  such that the fibration $(A_0 \times \tilde{S})/G \to \tilde{S}/G$ is birational to $\mathcal{A} \to \mathbf{P}^2$.
- The map $\tilde{S} \to \mathbf{P}^2$ is called the *trivializing base change*.

Diagram

$$\begin{tikzcd}
\tilde{S} \times A \arrow[r, dashed] \arrow[d] & \mathcal{A} \arrow[d] \\
\tilde{S} \arrow[r] & S \arrow[r] & \mathbf{P}^2
\end{tikzcd}$$

Albanese approach

- The points in $A(\mathbf{C}(s,t))$ correspond with *rational sections* $\mathbf{P}^2 \dashrightarrow \mathcal{A}$.
- The rational sections $\mathbf{P}^2 \dashrightarrow \mathcal{A}$ corresponds with G-equivariant rational sections $\tilde{S} \dashrightarrow \tilde{S} \times A_0$.
- The latter correspond with graphs of G-equivariant rational maps $\tilde{S} \dashrightarrow A_0$.
- Since the target is an abelian variety, we can extend each such a rational map to a morphism $\tilde{S} \to A_0$.
- Hence it sufficies to study the possible morphisms $\mathrm{Alb}(\tilde{S}) \to A_0$.
- **Difference with one-dimensional base variety:** $\mathrm{Alb}(\tilde{S})$ **is controlable!**

Diagram

$$\begin{tikzcd}
\tilde{S} \times A \arrow[r, dashed] \arrow[d, shift left] & \mathcal{A} \arrow[d, shift left] \\
\tilde{S} \arrow[r] \arrow[u, bend left] & S \arrow[r] & \mathbf{P}^2 \arrow[u, dashed, shift left]
\end{tikzcd}$$

Albanese approach

- **Difference with one-dimensional base variety:** $\mathrm{Alb}(\tilde{S})$ **is controlable!**
- Assume (for simplicity) that Δ is reduced.
- For each $p \in \Delta_{\mathrm{sing}}$ there is a so-called local Albanese variety Alb_p. (E.g., if p is a cusp then Alb_p is the $j = 0$ elliptic curve, if p is a node then $\mathrm{Alb}_p = 0$.)
- **Theorem (Libgober's local divisibility):** $\mathrm{Alb}(\tilde{S})$ is an isogeny factor of $\prod_{p \in \Delta_{\mathrm{sing}}} \mathrm{Alb}_p$.

Albanese approach

- **Theorem (Libgober's local divisibility):** $\mathrm{Alb}(\tilde{S})$ is an isogeny factor of $\prod_{p \in \Delta_{\mathrm{sing}}} \mathrm{Alb}_p$.
- In the case of $y^2 = x^3 + g(s,t,1)$, $\Delta = V(g)$ is a cuspidal plane curve we have that $\mathrm{Alb}_p = 0$ or $\mathrm{Alb}_p \cong E_0$.
- Hence $\mathrm{Alb}(\tilde{S})$ is isogeneous to a power of E_0.
- We obtain that the Mordell-Weil rank equals $2 \dim \mathrm{Alb}(\tilde{S})$.

Albanese approach (proof of C–L)

- **Theorem (Zariski–Libgober):** If G is cyclic then $\dim \mathrm{Alb}(\tilde{S})$ is effectively computable.
- Actually you find a formula in terms of defects of several very explicit linear systems.
- In the C–L case you find that $\dim \mathrm{Alb}(\tilde{S}))$ equals the defect of the linear system of degree $5/6 \deg(\Delta) - 3$ polynomials through Σ.

Albanese approach (application)

Theorem (Zariski)

Let $f \in \mathbf{C}[s,t]$ be an irreducible polynomial, $m = p^n$ be a prime power. Then the Albanese variety of the desingularization of the projective closure of

$$z^m = f(s,t)$$

is trivial.

Albanese approach (application)

Corollary

Let $f \in \mathbf{C}[s,t]$ be an irreducible polynomial, let $A, B \in \mathbf{C}$ be such that $4A^3 + 27B^2 \neq 0$. Let

- $E_1 : y^2 = x^3 + f^2$
- $E_2 : y^2 = x^3 + f^4$
- $E_3 : y^2 = x^3 + fx$
- $E_4 : y^2 = x^3 + Af^2x + Bf^3$

Then the rank of $E_i(\mathbf{C}(s,t))$ is zero.

Remark: The corollary holds also true if we replace $(\mathbf{C}[s,t], \mathbf{C}, \mathbf{C}(s,t))$ with $(K[t], K, K(t))$ and K is an algebraically closed field, but is false if K does not contain a third root of unity or a fourth root of unity.

Albanese approach

- Exploits that we can control both the dimension and the isogeny factors of the Albanese variety of the trivializing base change.
- Is very good to bound the Mordell-Weil rank.
- If you wan to determine the ranks then you have to put **few constraints on the general fiber A_0, but strong constaints on the singularities of the discriminant.**

Thom-Sebastiani in C-L case

- Cogolludo-Augustin–Libgober gave a formula for the Mordell-Weil rank of $y^2 = x^3 + g(s,t,1)$, with g a homogeneous polynomial such that the curve $\Delta := V(g)$ has only nodes and cusps.
- We are going to generalize this as follows. We take
  - a weighted homogeneous polynomial $f(x_1, x_2)$, with weights $w_1, w_2 \in \mathbf{Q}$ such that the weighted degree of f equals one and such that the curve $f = 0$ in $\mathbf{C}^2$ has an isolated singularity at the origin. (E.g., $f = x_1^3 - x_2^2$, $w_1 = 1/3, w_2 = 1/2$.)
  - a homogeneous polynomial $g(y_0, y_1, y_2) = 0$ such that the projective curve $V(g) \subset \mathbf{P}^2$ has isolated singularities.
- Let $H/\mathbf{C}(s,t)$ be the desingularization of the projective closure of the affine curve $f(x_1, x_2) + g(s,t,1) = 0$.
- We want to determine sufficient conditions on f, g such that we can compute the rank of $\mathrm{Jac}(H)(\mathbf{C}(s,t))$.

Affine Milnor fiber

- For the moment let $f \in \mathbf{C}[x_1, \ldots, x_n]$ be an arbitrary weighted homogeneous polynomial.
- Choose the weights $w_1, \ldots, w_n$ such that the degree of f equals 1.
- Denote with $F = \{f = 1\} \subset \mathbf{C}^n$ (smooth affine hypersurface). The affine Milnor fiber of f.
- For $\alpha \in \mathbf{C}$ let $\zeta(\alpha) := \exp(2\pi i \alpha)$.
- Let $T_f : \mathbf{C}^n \to \mathbf{C}^n$ be the automorphism mapping $x_i \to \zeta(w_i) x_i$.
- Then $T_f(F) = F$. Hence T_f induces a operator on $H^\bullet(F)$.

Cohomology of F

- Let s be the dimension of the singular locus of $f = 0$ in $\mathbf{C}^n$.
- Then $\tilde{H}^i(F) = 0$ for $i \geq n$ and $i \leq n - 2 - s$.
- So in our setup ($f(x_1, x_2) = 0$ has an isolated singularities and $g(y_0, y_1, y_2) = 0$ has one-dimensional singular locus) we have three intersecting groups $H^1(F), H^1(G), H^2(G)$.

$H^1(F)$

- We required that f has an isolated singularity. For a general weighted homogeneous *isolated* singularity in n variables there is a formula for the dimension of the $\zeta(\alpha)$ eigenspace of T_f acting on $H^{n-1}(F)$ in terms of the Jacobian ring of f.
- In our case: For $\alpha \in \mathbf{Q}$ set

$$\nu(\alpha) := \dim(\mathbf{C}[x_1, x_2]/(f_{x_1}, f_{x_2}))_{\alpha+1-w_1-w_2}$$

- Multiplicity of α in the Steenbrink spectrum of f.
- $\nu(\alpha)$ only depends on w_1, w_2 and α.
- Symmetry: $\nu(-\alpha) = \nu(\alpha)$.
- For $0 \leq \alpha < 1$ we have that $\dim H^{n-1}(F)_{\zeta(\alpha)}$ equals $\nu(\alpha) + \nu(\alpha - 1)$.

$H^1(G)$

- $H^1(G)$ depends on the singularities of C and their position.
- Libgober studied ideals of quasi-adjunction (multiplier ideals). He gave for any $0 < \alpha < 1$ an effective construnction of schemes $X^{(\alpha)} \subset \Delta_{\text{sing}}$ such that if

$$\delta_\alpha := \mathrm{length}(X^{(\alpha)}) - \dim \mathbf{C}[y_0, y_1, y_2]/(I(X^{(\alpha)}))_{\alpha d - 3}$$

Then

$$\dim H^1(G)_{\zeta(\alpha)} = \delta(\alpha) + \delta(1-\alpha)$$

- $\dim H^1(G)_{\zeta(0)}$ equals the number of irreducible components of Δ minus one.

$H^1(G)$

- Where does this come from:
- You may know Alexander polynomial of knots.
- Can also be defined for plane curves (Zariski).
- This is defined in terms of the fundamental group of $\mathbf{P}^2 \setminus \Delta$.
- Equivalent definition of the Alexander polynomial of G is the characteristic polynomial of T_g on $H^1(G)$.
- So the above tells you also how to compute the Alexander polynomial of a plane curve.

Main result

Theorem

Let $f \in \mathbf{C}[x_1, x_2]$ be a weighted homogeneous polynomial with rational weights w_1, w_2 and of weighted degree 1 and let $g \in \mathbf{C}[y_0, y_1, y_2]$ be a squarefree homogeneous polynomial of degree d. Assume

$$dw_1, dw_2 \in \mathbf{Z} \text{ and } \sum_{0 \leq \alpha < 1} \nu(\alpha)\delta_\alpha = 0.$$

Then the Mordell-Weil rank of the group of $\mathbf{C}(s,t)$-valued points of the Jacobian of the general fiber of $H : f(x,y) + g(s,t,1)$ equals

$$\sum_{0 < \alpha < 1} (\nu(\alpha) + \nu(\alpha - 1))(\delta(\alpha) + \delta(1 - \alpha))$$

Main result

If $f = x_1^2 + x_2^e$ and $V(g)$ is a curve with ADE singularities then

$$\sum_{0 \leq \alpha < 1} \nu(\alpha)\delta_\alpha = 0.$$

If $f = x_1^2 + x_2^3$ and $V(g)$ is a curve with nodes and ordinary cusps then you recover C–L.

Proof: reduction to a problem on affine Milnor fibers

- $W : Z(f(x_1, x_2) + g(y_0, y_1, y_2)) \subset \mathbf{P}(w_1, w_2, 1, 1, 1)$
- $H/\mathbf{C}(s, t)$ the smooth projective curve associated with $f(x_1, x_2) + g(s, t, 1)$.
- A variant of Shioda-Tate formula yields

$$\operatorname{rank} \operatorname{Jac}(H)(\mathbf{C}(s, t)) = \operatorname{rank} \mathrm{CH}^1(W) - \delta(0)\nu(0) - 1$$

Cycle class maps for singular varieties

- $\mathrm{CH}^1(W) \otimes \mathbf{Q}$ can be mapped injectively into $H^4(W, \mathbf{Q})$. (Lefschetz $(1,1)$ is unavailable, however every Weil divisor has a fundamental class inside $H_4(W, \mathbf{Z})$.)
- The image is sub-MHS of pure type $(2,2)$.
- If $H^4(W, \mathbf{Q})$ is of pure $(2,2)$ type then $\mathrm{CH}^1(W) \otimes \mathbf{Q} \cong H^4(W, \mathbf{Q})$.
- Let $F \oplus G := \{f + g = 1\} \subset \mathbf{C}^5$. Then there is a natural isomorphism of MHS

$$H^4(W, \mathbf{Q})_{\mathrm{prim}} \cong H^3(F \oplus G)^{T_{f+g}}$$

- We need to find out when $H^3(F \oplus G)$ is of pure $(2,2)$-type, and to have a way to calculate its dimension.

Thom-Sebastiani

- Advantage of working with Milnor fibers:
- Let $f \in \mathbf{C}[x_1, \ldots, x_n]$ and $g \in \mathbf{C}[y_1, \ldots, y_m]$ weighted homogeneous.
- Let $F := \{f = 1\}$, $G := \{g = 1\}$ and $F \oplus G := \{f + g = 1\}$.
- Thom-Sebastiani:

$$\tilde{H}^{\bullet+1}(F \oplus G) \cong \tilde{H}^{\bullet}(F) \otimes \tilde{H}^{\bullet}(G)$$

  and $T_{f+g} = T_f \otimes T_g$.
- There is no such results for $f = 0, g = 0$ and $f + g = 0$.
- This is not a morphism of MHS.

Thom-Sebastiani

- In our case ($f(x_1, x_2)$ has an isolated singularity, $g(y_0, y_1, y_2)$ one-dimensional singular locus) we find

$$H^3(F \oplus G)^{T_{f+g}} = \oplus_\alpha H^1(F)_{\zeta(\alpha)} \otimes H^1(G)_{\zeta(-\alpha)}$$

- Combining everything we get

$$h^4(W) = 1 + \sum_{0 \leq \alpha < 1} (\nu(\alpha) + \nu(\alpha - 1))(\delta(1 - \alpha) + \delta(\alpha))$$

- What about the MHS?

MHS

- The Thom-Sebastiani isomorphism

$$\tilde{H}^{\bullet+1}(F \oplus G) \cong \tilde{H}^{\bullet}(F) \otimes \tilde{H}^{\bullet}(G)$$

  is not an isomorphism of MHS.
- If $f = 0$ and $g = 0$ are arbitrary isolated singularities then Scherk and Steenbrink related the MHS on the LHS with the one on the RHS. (Tedious formulae.)
- Main point in the proof is now that we have similar formulae if $g = 0$ has a nonisolated singularity, but is weighted homogeneous.

MHS

- Upshot: $H^4(W)$ has a pure weight 4 MHS.
- $h^{4,0}$ and $h^{0,4}$ vanish by dimension reasons.
- $h^{3,1}$ equals
$$\sum_{0<\alpha<1} \delta(\alpha)\nu(\alpha)$$
- And this vanihes by assumption. Hence
$h^4(W) = \operatorname{rank} \operatorname{Jac}(H)(\mathbf{C}(s,t)) + \nu(0)\alpha(0) + 1$

$$\operatorname{rank} \operatorname{Jac}(H)(\mathbf{C}(s,t)) = \sum_{0<\alpha<1} (\nu(\alpha)+\nu(\alpha-1))(\delta(1-\alpha)+\delta(\alpha))$$

Toric decompositions

- Let $g \in \mathbf{C}[y_0, y_1, y_2]$ be a squarefree homogeneous polynomial.
- A (p, q)-toric decomposition consits of homogeneous polynomials h_1, h_2 such that

$$h_1^p + h_2^q = g$$

- A (p, q)-quasi-toric decomposition consits of homogeneous polynomials h_1, h_2, h_3 such that

$$h_1^p + h_2^q = h_3^r g$$

  with $r = \mathrm{lcm}(p, q)$.
- Equivalently, these decompositions correspond with $\mathbf{C}[s, t]$ and $\mathbf{C}(s, t)$ points on the curve

$$H_{p,q} : x_1^p + x_2^q - g(s, t, 1) = 0$$

Toric decompositions

- If $(p, q) \in \{(2, 3), (3, 3), (2, 4)\}$ then $H_{p,q} \cong \mathrm{Jac}(H_{p,q})$.
- The quasi-toric decompositions form an abelian group and we can often calculate its rank.
- Extra structure: we can introduce a height pairing on the quasi-toric decompositions. (Shioda's constructions of the Mordell-Weil lattice.)

Toric decompositions

- From now on $(p, q) = (2, 3)$, $H = H_{2,3}$ and $k = \lceil \deg(g)/6 \rceil$.
- In our case the Mordell-Weil lattice is an integral lattice.
- The nonzero vectors have length at least $2k$. (g squarefree.)
- The vectors of length $2k$ correspond with the toric decomposition.

Irreducible sextics

- Let g be an irreducible homogeneous polynomial of degree 6.
- Then $\delta(\alpha) = 0$ for $\alpha \neq 5/6$. (Zariski's theorem excludes $1/3, 1/2, 2/3$, semicontinuity of the spectrum excludes $1/6$.)
- Degtyarev: Let C be a plane sextic. Let t be the number of toric decompositions of C. Then $(\delta(5/6), t) \in \{(0,0), (1,6), (2,24), (3,72)\}$.
- Proof uses a lot of properties of the $K3$-surface $z^2 = g(y_0, y_1, y_2)$.

Irreducible sextics

- The toric decompositions generate a root lattice ($k = 1$) contained in the Mordell-Weil lattice (which has rank $2\delta(5/6)$).
- Classification of roots lattices it follows that in each case there is only one possible lattice.
- The toric decompositions generate (depending on $\delta(5/6)$) the lattice $0, A_2, D_4, E_6$.
- In particular, $H(\mathbf{C}(s,t))$ is generated by $H(\mathbf{C}[s,t])$.

Zariski pairs

- Aim: find a $g \in \mathbf{C}[y_0, y_1, y_2]_{6n}$ such that the Morell-Weil group is generated (as $\mathbf{Z}[\omega]$-module) by a point of the form

$$\left(\frac{h_0}{l^2}, \frac{h_1}{l^3}\right)$$

  with $\deg(l) = 1, \deg(h_0) = 6, \deg(h_1) = 9$.
- If $n = 1$ then the Mordell-Weil rank equals 4.
- If $n = 2$ then one can find such a g with 30 cusps.
- There is also a degree 12 cuspidal curve with 30 cusps, such that the Mordell-Weil group is generated by a point of the form (h_2, h_3) with $\deg(h_2) = 4, \deg(h_3) = 6$
- They yield distinct Mordell-Weil lattices.

Zariski pairs

- Can shouw that (for this type of curves) the Mordell-Weil lattices is a deformation invariant.
- So the locus of the degree 12 curves with 30 cusps has at least 2 irreducible components.
- These two curves form a so-called (Alexander-equivalent) Zariski pair.